



6.8 Bases and dimension

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Introduction

- In this topic, we will
 - Define a basis
 - Show how to find a basis from a collection of vectors
 - See that if the span of two bases are equal,
both bases must have the same number of vectors
 - Define the dimension of a vector space or subspace
 - Discuss the benefits of having an orthonormal basis





Definition

- To this point, we have simply interpreted a vector as being coefficients of particular values

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

- An alternative interpretation is, however, that these coefficients are coefficients in a linear combination of basis vectors:

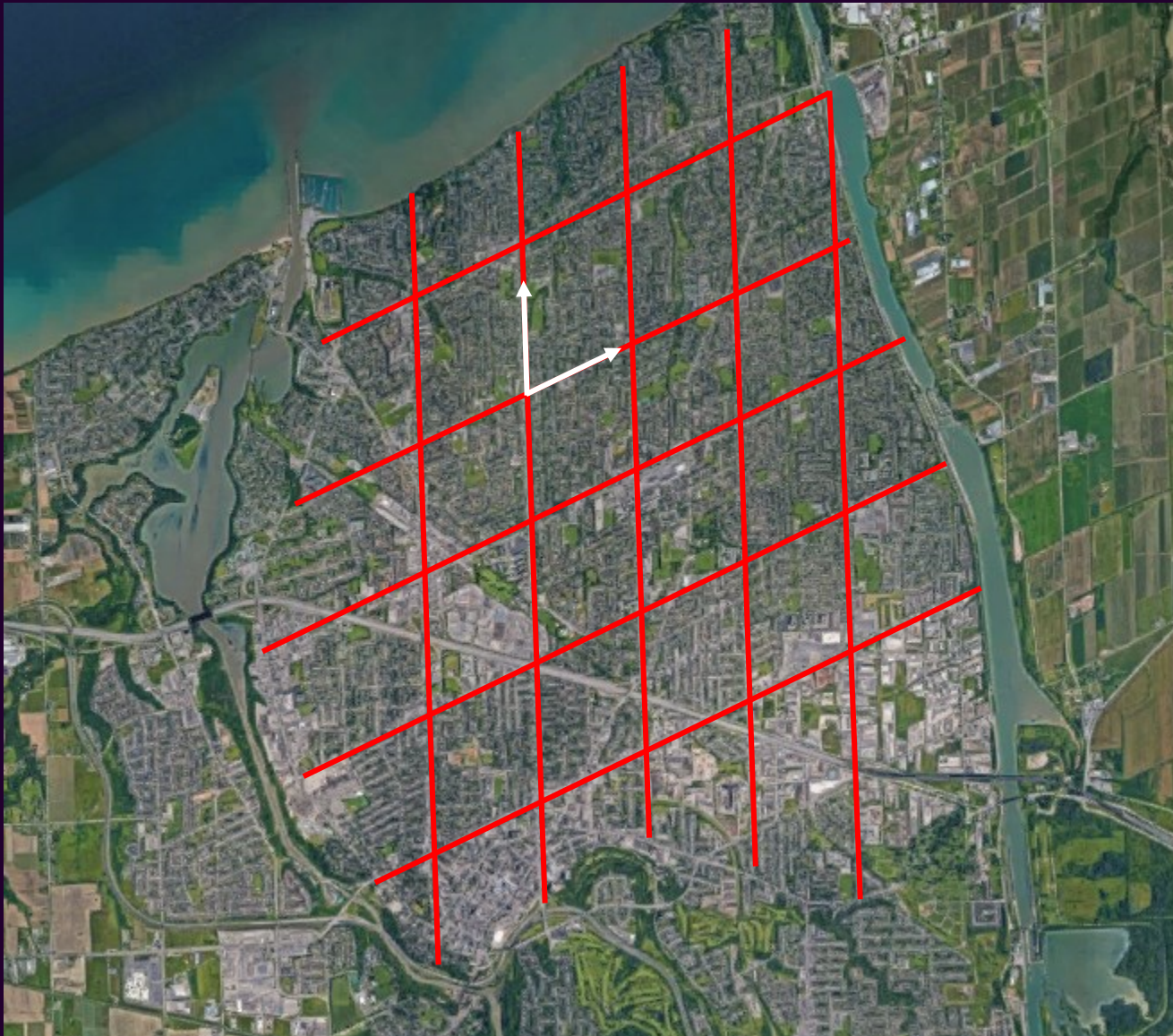
$$\mathbf{u} = u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Move u_1 times the length of the first canonical vector in that direction, move u_2 times the length of the second canonical vector in that direction, and so on...





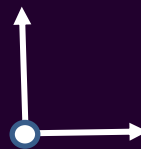
Consider St. Catharines, Ontario





Finding a basis

- Also, consider you are dealing with drones and a search-and-rescue
 - You know that the objective you are searching for is 154 m west of you and 295 m north
 - These are coordinates with respect to the canonical basis
 - Your vehicle, helicopter, or drone may, however, be facing in a different direction:
 - You would like to know where the object is relative to your front and your right





Example

- Thus, given a

$$\begin{pmatrix} -1.2 \\ 4.8 \\ 6 \\ -5.4 \end{pmatrix}, \begin{pmatrix} -2.8 \\ -1.8 \\ 4 \\ -5.6 \end{pmatrix}, \begin{pmatrix} -12 \\ 0.2 \\ 4 \\ 2.2 \end{pmatrix}, \begin{pmatrix} -9.6 \\ -15.6 \\ -7 \\ 7.2 \end{pmatrix}$$

$$\begin{pmatrix} -1.2 & -2.8 & -12 & -9.6 \\ 4.8 & -1.8 & 0.2 & -15.6 \\ 6 & 4 & 4 & -7 \\ -5.4 & -5.6 & 2.2 & 7.2 \end{pmatrix} \sim \begin{pmatrix} 6 & 4 & 4 & -7 \\ 0 & -5 & -3 & -10 \\ 0 & 0 & -10 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$





Example

- Do these form a basis, and if not, find a subset that does:

$$\begin{pmatrix} 2.8 \\ 4 \\ -2.4 \\ 3.6 \end{pmatrix}, \begin{pmatrix} 2.1 \\ 3 \\ -1.8 \\ 2.7 \end{pmatrix}, \begin{pmatrix} 1.4 \\ 2 \\ -1.2 \\ 1.8 \end{pmatrix}, \begin{pmatrix} -8.4 \\ -12 \\ 4.2 \\ -12.6 \end{pmatrix}$$

- We note that

$$\begin{pmatrix} 2.8 & 2.1 & 1.4 & -8.4 \\ 4 & 3 & 2 & -12 \\ -2.4 & -1.8 & -1.2 & 4.2 \\ 3.6 & 2.7 & 1.8 & -12.6 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 2 & -12 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, a basis for the span is Vectors 1 and 4





Example

- Do these form a basis, and if not, find a subset that does:

$$\begin{pmatrix} 0 \\ -0.5 \\ 5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 8.8 \\ -8.3 \\ -5 \\ 11.5 \end{pmatrix}, \begin{pmatrix} -6 \\ 1.8 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6.4 \\ -12.9 \\ 5 \\ 7.5 \end{pmatrix}$$

- We note that

$$\begin{pmatrix} 0 & 8.8 & -6 & 6.4 \\ -0.5 & -8.3 & 1.8 & -12.9 \\ 5 & -5 & 0 & 5 \\ -0.5 & 11.5 & 0 & 7.5 \end{pmatrix} \sim \begin{pmatrix} 5 & -5 & 0 & 5 \\ 0 & 11 & 0 & 8 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

- Thus, this is a basis for the span





Example

- In $\mathbf{R}^m$, the m vectors are always a basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- After all, the rank of the corresponding matrix is m

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$





Observations

Theorem

Given two basis of a vector space that is either $\mathbf{F}^m$ or a subspace thereof,

they must have the same number of vectors.

A sketch of a proof:

Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are a basis for a vector space, and $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ is a second basis

$$\begin{pmatrix} u_{1,1} & u_{2,1} & \vdots & v_{1,1} & v_{2,1} & v_{3,1} \\ u_{1,2} & u_{2,2} & \vdots & v_{1,2} & v_{2,2} & v_{3,2} \\ u_{1,3} & u_{2,3} & \vdots & v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \sim \begin{pmatrix} \square & ? & \vdots & ? & ? & ? \\ 0 & \square & \vdots & ? & ? & ? \\ 0 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix}$$

This must be true, as we must be able to write each of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$





Observations

However, if we performed those exact same row operations on the matrix

$$\begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \sim \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ 0 & 0 & 0 \end{pmatrix}$$

No matter what these values are,
the maximum rank is 2

- Thus, they are not linearly independent

If they are a basis, however, they must be linearly independent

- This leads to a contradiction, and thus both bases must have had three vectors ■





Observations

- Also, every different linear combination of basis vectors defines a unique vector in the span
- Identically, every vector in the span of a basis must have a unique linear combination of basis vectors that equal it
 - If $\mathbf{u}_1, \dots, \mathbf{u}_n$ is a basis, then given an arbitrary vector $\mathbf{v}$, the system of linear equations defined by

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{v}$$

either has no solutions ($\mathbf{v}$ is not in the span)
or it has one unique solution





Definition

Definition

The *dimension* of the vector space that is the span of n vectors is said to have a dimension of n

- We will describe such a vector space to be an n -dimensional vector space
- If the span of these vectors is a subspace,
we will describe the subspace as being n -dimensional





Example

- For example, every vector in $\mathbf{R}^3$ can be written as a linear combination of the three linearly independent vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Thus, the span of these vectors is $\mathbf{R}^3$ and thus these form a basis for $\mathbf{R}^3$, and the dimension of $\mathbf{R}^3$ is three
 - Thus, every basis for $\mathbf{R}^3$ must have three vectors
 - Thus, every collection of three linearly independent vectors in $\mathbf{R}^3$ is a basis for $\mathbf{R}^3$
 - This particular basis is called the *canonical basis* for $\mathbf{R}^3$





Example

- For example, the following two vectors in $\mathbf{R}^3$ are linearly independent:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$$

- Because they are independent, they form a basis for their span
- Because there are two vectors,
dimension of the span of these two vectors is two
 - The span is a plane in $\mathbf{R}^3$ that passes through the origin
- Because there are only two vectors in this basis,
the span cannot be all of $\mathbf{R}^3$
 - It is a subspace of $\mathbf{R}^3$





Example

- For example, the a single non-zero $\mathbf{R}^3$ is linearly independent:

$$\begin{pmatrix} -1 \\ -6 \\ -7 \end{pmatrix}$$

- Because it is independent, it forms a basis for its span
 - That is, it is a basis for all scalar multiples of it
- Because there is one vector,
the dimension of the span of this vectors is one
 - The span is a line in $\mathbf{R}^3$ that passes through the origin
- Because there is only one vector in this basis,
the span cannot be all of $\mathbf{R}^3$





Bases for $\mathbf{R}^m$ and $\mathbf{C}^m$

- We have already seen that these m vectors in $\mathbf{F}^m$ are linearly independent and therefore form a basis of an m -dimensional vector space

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- However, every vector in $\mathbf{F}^m$ can be written as a linear combination of these vectors; thus, the span is all of $\mathbf{F}^m$
- Thus, all bases of $\mathbf{F}^m$ must have m vectors,
and any collection of m linearly independent vectors forms a basis
- This basis is, however, called the *canonical* basis for $\mathbf{F}^m$
- The span of any collection of n linearly independent vectors where $n < m$ is a basis for an n -dimensional subspace of $\mathbf{F}^m$





Orthogonal and orthonormal bases

Definition

If the vectors in a basis are all mutually orthogonal, that basis is described as being an *orthogonal basis*

Definition

If the vectors in an orthogonal basis are also normalized, that basis is described as being an *orthonormal basis*





Orthogonal and orthonormal bases

- Claim

- Given a basis of n vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$, if we perform the Gram Schmidt algorithm on these n vectors to get n orthonormal vectors $\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n$, then

$$\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\} = \text{span}\{\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n\}$$

- Because each original vector can be written as a linear combination of the orthonormal vectors, and because each of the orthonormal vectors is a linear combination of the original vectors,
their spans must overlap
- But because they are both bases with the same number of vectors, these spans must be identical





Orthogonal and orthonormal bases

- The clearest benefit of an orthonormal basis is as follows:
 - If $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$,
we must solve a system of linear equations in order to find the coefficients:

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n$$

- If no solution exists to this system,
then $\mathbf{v} \notin \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$
- Now, given an orthonormal basis,
 - Given any $\mathbf{v}$, $\tilde{\mathbf{v}} = \langle \hat{\mathbf{u}}_1, \mathbf{v} \rangle \hat{\mathbf{u}}_1 + \dots + \langle \hat{\mathbf{u}}_n, \mathbf{v} \rangle \hat{\mathbf{u}}_n$ is the best approximation of $\mathbf{v}$ in the span of the orthonormal basis
 - Also, $\tilde{\mathbf{v}} = \mathbf{v}$ if and only if $\mathbf{v} \in \text{span}\{\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n\}$





Summary

- Following this topic, you now
 - Understand the idea of a basis for a vector space or subspace
 - Can take a collection of vectors and find a subset that forms a basis for the span of the original collection of vectors
 - Know that all bases for a vector space or subspace must have the same number of vectors, at least for finite dimensions
 - Are aware that orthonormal bases are better than arbitrary bases, and that it is much easier to work with orthonormal bases
 - Understand that the span of a bases equals the span of the orthonormal vectors that result from the Gram-Schmidt algorithm





References

- [1] https://en.wikipedia.org/wiki/Linear_independence
- [2] https://en.wikipedia.org/wiki/Linear_span
- [3] [https://en.wikipedia.org/wiki/Basis_\(linear_algebra\)](https://en.wikipedia.org/wiki/Basis_(linear_algebra))
- [4] https://en.wikipedia.org/wiki/Gram-Schmidt_process





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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